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Applied Mathematics Letters 15 (2002) 709–720

Applied Mathematics Letters

www.elsevier.com/locate/aml

Constant-Gain Observer for a Class of Multi-Output Nonlinear Systems

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(Received July 2000; revised and accepted July 2001)

Abstract—A new approach to design a constant-high gain exponential observer for triangular multi-output nonlinear systems is proposed. This observer is designed to systems satisfying some regularity assumptions.

Keywords—Nonlinear systems, Multi-output systems, Triangular systems, Constant high-gain observer.

1. INTRODUCTION

In order to apply advanced concepts of control and diagnosis to practical applications, the knowledge of state variables is often required. This can be achieved by means of state observers.

This paper deals with the design of observers for a special class of multi-output nonlinear systems satisfying some regularity assumptions. The general framework of this observer design is based on the works of [1–4].

In [2], the authors have designed an observer for single output control affine systems which are observable for every inputs (uniformly observable). It is shown in [5] (for a short proof see [2]) that single output control affine systems which are observable for every inputs can be transformed locally almost everywhere into a canonical observable form. This canonical form is composed of a fixed linear dynamics component and a triangular controlled one. Using this canonical form, the authors in [2] have designed a high gain observer for such systems under

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0893-9659/02/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
PII: S0893-9659(02)00032-0

some global Lipschitz assumptions on the controlled part. The gain of the proposed observer is issued from an algebraic Lyapunov equation. An extension of this observer synthesis to the multi-output case is given in [1,3,6] for a larger class of systems which are observable for every inputs. More precisely, the authors have considered uniformly observable multi-output systems which are split into a (time-varying) linear part and a nonlinear controlled part. Under some structure assumptions on the controlled part, the authors have shown that the technique involved in [2] can be generalized for such systems. A further extension of this last result for systems with locally regular inputs is given in [1]. But here the observer gain involves the integration of some differential Lyapunov equations. This last work in particular generalizes those of [1,2,7].

Unlike the just cited works where the gains of the proposed observers were time-varying, the authors in [4] proposed a constant gain observer for general single output systems which are uniformly infinitesimally observable under some regularity assumptions on the vector fields. The practical construction of the proposed observer is difficult to realize in the sense that the computation of the observer gain is not direct.

In the present paper, we use the general strategy of observer design adopted in [4] to construct a constant-high gain observer for multi-output nonlinear triangular systems under similar regularity assumptions. Indeed, we first propose a new constant-gain observer for single output systems. The main difference between this observer and that proposed by Gauthier and Kupka lies in the simplicity of the former. Indeed, a systematic procedure which allows the construction of the observer gain is presented. The extension of the observer synthesis to the multi-output case is then treated and again a systematic procedure allowing the gain construction is given.

Our work is organized as follows. Section 2 is devoted to observer design in the single-output case which will appear as a particular case of the multi-output one. However, we prefer to present it separately for a better understanding of our results. In Section 3, multi-output nonlinear systems are considered and corresponding observers are proposed. Proofs of theorems are given in this section.

2. OBSERVER DESIGN FOR SINGLE-OUTPUT NONLINEAR SYSTEMS

Consider the single-output nonlinear system of the following triangular form:

$$\begin{aligned}
\dot{x}_1 &= f_1(u, x_1, x_2),\\
\dot{x}_2 &= f_2(u, x_1, x_2, x_3),\\
&\ \vdots\\
\dot{x}_{n-1} &= f_{n-1}(u, x_1, \dots, x_n),\\
\dot{x}_n &= f_n(u, x_1, \dots, x_n),\\
y &= x_1,
\end{aligned} \tag{1}$$

where the input $u(t) \in U$ a compact subset of $\mathbb{R}^m$, the output $y \in \mathbb{R}$, and f_k; $k = 1, \dots, n$ are functions of class C^r ($r \geq 1$) with respect to their arguments. The more condensed form of (1) is

$$\begin{aligned}
\dot{x} &= f(u, x),\\
y &= Cx,
\end{aligned} \tag{2}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n; \qquad f = [f_1, \dots, f_n]^\top, \qquad \text{and} \qquad C = [1, 0, \dots, 0].$$

Set

$$a_k(t) \triangleq a_k(u, x_1, \dots, x_{k+1}) = \frac{\partial f_k}{\partial x_{k+1}}(u, x_1, \dots, x_{k+1}); \qquad k = 1, \dots, n-1. \tag{3}$$

In the sequel, we are going to design a high gain observer for system (2). As in [1–4], the design of such an observer requires some additional assumptions. Indeed, we shall assume the following.

(A1) The functions f_k; $k = 1, \dots, n$ are global Lipschitz w.r.t. x:

$$\exists \beta > 0; \qquad \forall (x, u) \in \mathbb{R}^n \times U, \qquad \left\| \frac{\partial f_k}{\partial x}(u, x) \right\| \le \beta.$$

(A2) There exist two constants $0 < \bar{\alpha} < \bar{\beta} < +\infty$, s.t. $\forall (x, u) \in R^n \times U$, we have

$$0 < \bar{\alpha} \le a_k(t) \triangleq \frac{\partial f_k}{\partial x_{k+1}}(u, x) \le \bar{\beta}; \qquad k = 1, \dots, n-1. \tag{4}$$

In the sequel, we will use the following notations:

- $$\mathcal{A}_k(t) = \begin{bmatrix} 0 & a_1(t) & 0 & & 0 \\ \vdots & & a_2(t) & & \\ 0 & & & \ddots & a_{k-1}(t) \\ 0 & \dots & & 0 & 0 \end{bmatrix}; \qquad k = 2, \dots, n, \tag{5}$$

where, the a_i's; $i = 1, \dots, k-1$ are defined as in (3) and satisfy (4).

- $C_k = [1, 0 \dots, 0]$ is the k line vector.

- $$\mathcal{S}_k = \begin{bmatrix} s_{11} & s_1 & 0 & & 0 \\ s_1 & s_{22} & \ddots & & \vdots \\ 0 & \ddots & & \ddots & 0 \\ \vdots & & & \ddots & s_{k-1} \\ 0 & \dots & 0 & s_{k-1} & s_{kk} \end{bmatrix}; \qquad k = 2, \dots, n, \tag{6}$$

where all the terms are constant and are such that $s_{ii} > 0$ and $s_j < 0$; $(i = 1, \dots, k;\ j = 1, \dots, k-1)$.

Before stating our main theorem, we need the following lemma.

LEMMA. *There exists $\mathcal{S}_n$ an $n \times n$ symmetric positive definite (S.P.D.) constant matrix of the form (6); $\exists \eta_n > 0$ such that*

$$\mathcal{A}_n^\top(t)\mathcal{S}_n + \mathcal{S}_n\mathcal{A}_n(t) - C_n^\top C_n \le -\eta_n I_n, \qquad \forall t \ge 0, \tag{7}$$

where $\mathcal{A}_n(t)$ is given by equation (5) and I_n is the $n \times n$ identity matrix.

PROOF OF LEMMA. we will proceed by induction.

Set

$$Q_k(t) = \mathcal{A}_k^\top(t)\mathcal{S}_k + \mathcal{S}_k\mathcal{A}_k(t) - C_k^\top C_k, \qquad \text{for } k = 2, \dots, n.$$

For $k = 2$, a simple calculation gives:

$$Q_2 = \begin{pmatrix} -1 & s_{11}a_1(t) \\ s_{11}a_1(t) & 2s_1a_1(t) \end{pmatrix}.$$

Now, let $s_{11} > 0$ and $s_1 < 0$ be two arbitrary constants and let $x = \binom{x_1}{x_2}$; $x_1, x_2 \in \mathbb{R}$, we obtain:

$$\begin{aligned} x^\top Q_2 x &= -x_1^2 + 2s_1a_1(t)x_2^2 + 2s_{11}a_1(t)x_1x_2 \\ &\le -x_1^2 - 2\bar{\alpha}|s_1|x_2^2 + 2\bar{\beta}s_{11}|x_1||x_2|, \end{aligned}$$

where $\bar{\alpha}, \bar{\beta}$ come from (A2).

Set $\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$ where $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2\sqrt{2\bar{\alpha}|s_1|}$. We obtain

$$\begin{aligned}
\tilde{x}^\top Q_2 \tilde{x} &\leq -\tilde{x}_1^2 - \tilde{x}_2^2 + 2\,\frac{\bar{\beta}s_{11}}{\sqrt{2\bar{\alpha}|s_1|}}\,|\tilde{x}_1|\,|\tilde{x}_2| \\
&\leq -\left(1 - \frac{\bar{\beta}s_{11}}{\sqrt{2\bar{\alpha}|s_1|}}\right)\|\tilde{x}\|^2 \\
&\leq -\eta_2\,\|\tilde{x}\|^2,
\end{aligned}$$

where $\eta_2 = (1 - \bar{\beta}s_{11}/\sqrt{2\bar{\alpha}|s_1|}) > 0$ as soon as $|s_1| > \bar{\beta}^2 s_{11}^2/2\bar{\alpha}$.

Finally, it suffices to choose s_{22} such that the matrix $\mathcal{S}_2$ is S.P.D. (take for example $s_{22} > s_1^2/s_{11}$).

Now, assume that (7) holds for $k = n-1$. It means that there exists a constant S.P.D. matrix $\mathcal{S}_{n-1}$ s.t. for every trajectory of (2), we have:

$$\mathcal{A}_{n-1}^\top(t)\mathcal{S}_{n-1} + \mathcal{S}_{n-1}\mathcal{A}_{n-1}(t) - C_{n-1}^\top C_{n-1} \leq -\eta_{n-1}I_{n-1}, \qquad \forall t \geq 0,$$

where $\eta_{n-1} > 0$ is constant. Let us show (7) for n. Consider the following matrix:

$$\mathcal{S}_n = \begin{pmatrix} & & & 0 \\ & & & \vdots \\ & \mathcal{S}_{n-1} & & 0 \\ & & & s_{n-1} \\ 0 \;\dots & 0 & s_{n-1} & s_{nn} \end{pmatrix},$$

where $s_{nn} > 0$ is a constant which will be specified later, and $\mathcal{S}_{n-1}$ is given by the induction hypothesis.

Set

$$Q_n = \mathcal{A}_n^\top \mathcal{S}_n + \mathcal{S}_n \mathcal{A}_n - C_n^\top C_n.$$

A simple calculation shows that:

$$Q_n = \begin{pmatrix} & & & & 0 \\ & Q_{n-1} & & & \vdots \\ & & & & 0 \\ & & & & s_{n-2}a_{n-1} \\ & & & & s_{n-1,n-1}a_{n-1} \\ 0 \;\dots & 0 & s_{n-2}a_{n-1} & s_{n-1,n-1}a_{n-1} & 2s_{n-1}a_{n-1} \end{pmatrix},$$

where

$$Q_{n-1} = \mathcal{A}_{n-1}^\top(t)\mathcal{S}_{n-1} + \mathcal{S}_{n-1}\mathcal{A}_{n-1}(t) - C_{n-1}^\top C_{n-1}.$$

Set $x = \begin{pmatrix} X^{n-1} \\ x_n \end{pmatrix}$, where $X^{n-1} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$, $x_i \in \mathbb{R}$, for $i = 1, \dots, n$, and let $s_{n-1} < 0$, we get

$$\begin{aligned}
x^\top Q_n x &= -x_1^2 + 2X^{n-1T}\mathcal{A}_{n-1}^\top(t)\mathcal{S}_{n-1}X^{n-1} + 2s_{n-1}a_{n-1}(t)x_n^2 \\
&\quad + 2s_{n-2}a_{n-1}(t)x_{n-2}x_n + 2s_{n-1,n-1}a_{n-1}(t)x_{n-1}x_n \\
&\leq -\eta_{n-1}\left\|X^{n-1}\right\|^2 - 2\bar{\alpha}|s_{n-1}|x_n^2 + 2\bar{\beta}|s_{n-2}||x_{n-2}||x_n| + 2\bar{\beta}s_{n-1,n-1}|x_{n-1}||x_n|.
\end{aligned}$$

Thus,

$$x^\top Q_n x \leq -\eta_{n-1}\left\|X^{n-1}\right\|^2 - 2\bar{\alpha}|s_{n-1}|x_n^2 + 2\bar{\beta}s\left\|X^{n-1}\right\||x_n|,$$

where $s = \max(|s_{n-2}|, s_{n-1,n-1})$.

Proceeding as above, it is easy to see that for $|s_{n-1}|$ high enough, there exists η_n such that

$$x^{\top} Q_n x \leq -\eta_n \|x\|^2.$$

Finally and similarly as above, we can choose $s_{nn} > 0$ such that $\mathcal{S}_n$ remains S.P.D.

This ends the proof of the lemma. ∎

Our candidate observer takes the following form:

$$\dot{\hat{x}} = f(u, \hat{x}) - \theta \Delta_\theta \mathcal{S}^{-1} C^{\top} (C\hat{x} - y). \tag{10}$$

Here $\mathcal{S} \triangleq \mathcal{S}_n$ is given by the lemma, $C \triangleq C_n = [1, 0, \ldots, 0]$, $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{bmatrix} \in \mathbb{R}^n$, u and y are, respectively, the input and the output of system (2), $\Delta_\theta = \operatorname{diag}(1, \theta, \theta^2, \ldots, \theta^{n-1})$ for some $\theta > 0$.

We then state the following.

THEOREM 1. *Under Assumptions (A1) and (A2), there exists* $\theta_0 > 0$ *such that*

$$\forall \theta > \theta_0; \quad \forall u \in U; \quad \exists \lambda_\theta > 0; \quad \exists \mu_\theta > 0; \quad \forall x(0) \in \mathbb{R}^n; \quad \forall \hat{x}(0) \in \mathbb{R}^n;$$

$$\|\hat{x}(t) - x(t)\| \leq \lambda_\theta e^{-\mu_\theta t} \|\hat{x}(0) - x(0)\|,$$

where $x(t)$ *is the trajectory of (2) associated to the initial state* $x(0)$ *and the input* u, $\hat{x}(t)$ *is any trajectory of system (10) with input* u *and output* y.

Instead of giving the proof of Theorem 1, we shall give it in the multi-output case which includes systems described by (2).

3. EXTENSION TO MULTI-OUTPUT CASE

Consider the multi-output nonlinear system of the following form:

$$\begin{aligned}
\dot{x}_1^1 &= f_1^1(u, x^1, x_1^2), \\
\dot{x}_2^1 &= f_2^1(u, x^1, x_1^2, x_2^2), \\
&\vdots \\
\dot{x}_p^1 &= f_p^1(u, x^1, x^2), \\
\dot{x}_1^2 &= f_1^2(u, x^1, x^2, x_1^3), \\
\dot{x}_2^2 &= f_2^2(u, x^1, x^2, x_1^3, x_2^3), \\
&\vdots \\
\dot{x}_p^2 &= f_p^2(u, x^1, x^2, x^3), \\
&\vdots \\
&\vdots \\
\dot{x}_1^n &= f_1^n(u, x), \\
\dot{x}_2^n &= f_2^n(u, x), \\
&\vdots \\
\dot{x}_p^n &= f_p^n(u, x), \\
y &= x^1,
\end{aligned} \tag{11}$$

where

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \in \mathbb{R}^{pn}; \qquad x^k = \begin{bmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_p^k \end{bmatrix} \in \mathbb{R}^p; \qquad k = 1, \dots, n$$

the input $u(t) \in U$ a compact subset of $\mathbb{R}^m$, the output $y \in \mathbb{R}^p$, and $f_i^k (k = 1, \dots, n;\ i = 1, \dots, p)$ are functions of class C^r $(r \geq 1)$ with respect to their arguments.

System (11) can be written in the following triangular condensed form:

$$\begin{aligned} \dot{x}^1 &= f^1\left(u, x^1, x^2\right), \\ \dot{x}^2 &= f^2\left(u, x^1, x^2, x^3\right), \\ &\vdots \\ \dot{x}^n &= f^n(u, x), \\ y &= Cx, \end{aligned} \tag{12}$$

where,

$$f^k = \begin{bmatrix} f_1^k \\ f_2^k \\ \vdots \\ f_p^k \end{bmatrix}; \qquad k = 1, \dots, n; \qquad C = [I_p, \dots, 0],$$

with I_p the $p \times p$ identity matrix.

A more condensed form of (11) is

$$\begin{aligned} \dot{x} &= F(u, x), \\ y &= Cx, \end{aligned} \tag{13}$$

where

$$F = \begin{bmatrix} f^1 \\ f^2 \\ \vdots \\ f^n \end{bmatrix}.$$

Set

$$A_k\left(u, x^1, \dots, x^{k+1}\right) = \frac{\partial f^k}{\partial x^{k+1}}\left(u, x^1, \dots, x^{k+1}\right); \qquad k = 1, \dots, n-1. \tag{14}$$

Along trajectories of (13), A_k takes the following form:

$$A_k(t) = \begin{bmatrix} a_{11}^k(t) & 0 & & \dots & 0 \\ a_{21}^k(t) & a_{22}^k(t) & 0 & \dots & \vdots \\ \vdots & & & & 0 \\ a_{p1}^k(t) & \dots & & & a_{pp}^k(t) \end{bmatrix}; \qquad k = 1, \dots, n-1, \tag{15}$$

with

$$a_{lj}^k(t) \triangleq \frac{\partial f_l^k}{\partial x_j^{k+1}}(u(t), x(t)); \qquad k = 1, \dots, n-1; \quad l = 1, \dots, p; \quad j = 1, \dots, l.$$

In the sequel, we are going to design a high gain observer for system (13). As in Section 1, the design of such an observer requires some additional assumptions. Indeed, we shall assume the following.

(A′1) The functions $f_i^k; (i = 1, \dots, p;\ k = 1, \dots, n)$ are global Lipschitz w.r.t. x:

$$\exists\, \beta > 0; \qquad \forall\, (u, x) \in \mathbb{R}^{pn} \times U, \qquad \left\| \frac{\partial f_i^k}{\partial x}(x, u) \right\| \leq \beta.$$

(A′2) There exist two constants $0 < \bar{\alpha} < \bar{\beta} < +\infty$, s.t. $\forall\, (x, u) \in R^{pn} \times U$, we have:

$$0 < \bar{\alpha} \le a_{ii}^k(t) \triangleq \frac{\partial f_i^k}{\partial x_i^{k+1}}\,(u, x) \le \bar{\beta}; \qquad k = 1, \dots, n-1; \quad i = 1, \dots, p. \tag{16}$$

REMARK. According to Assumptions (A′1), (A′2), we have

$$\operatorname{rank}(A_k(t)) = p; \qquad k = 1, \dots, n-1, \tag{17}$$

and for every trajectory of (13), we have

$$\sup_{t \ge 0} \|A_k(t)\| \le \beta; \qquad k = 1, \dots, n-1. \tag{18}$$

Before stating our main theorem, we need some technical results which we will present under the form of two lemmas.

LEMMA 1. *Under Assumptions (A′1), (A′2), and for each $k = 1, \dots, n-1$; there exists a $p \times p$ diagonal negative definite matrix $S_k = \operatorname{diag}(\lambda_1^k, \dots, \lambda_p^k)$; there exists a constant $\mu_k > 0$ such that*

$$S_k A_k(t) + A_k^\top(t) S_k \le -\mu_k I_p, \tag{19}$$

where I_p is the $p \times p$ identity matrix. Moreover, S_k and μ_k only depend on $\bar{\alpha}, \bar{\beta}$ and the Lipschitz constants of the f_i^k.

PROOF OF LEMMA 1. Denote by $P(t, z)$ the quadratic form

$$P(t, z) = z^\top \left(A_k^\top(t) S_k + S_k A_k(t)\right) z, \qquad z \in \mathbb{R}^p.$$

We obtain

$$\begin{aligned} P(t, z) &= -2 \sum_{i=1}^{p} |\lambda_i^k| a_{ii}^k(t) z_i^2 + 2 \sum_{i=2}^{p} \sum_{j=1}^{i-1} \lambda_i^k a_{ij}^k(t) z_i z_j \\ &\le -2\bar{\alpha} \sum_{i=1}^{p} |\lambda_i^k| z_i^2 + 2\beta \sum_{i=2}^{p} \sum_{j=1}^{i-1} |\lambda_i^k| |z_i| |z_j|, \end{aligned}$$

where β and $\bar{\alpha}$ are respectively given in Assumptions (A′1) and (A′2).

Set $\tilde{z}_i = z_i \sqrt{2|\lambda_i^k| \bar{\alpha}}$, then,

$$P(t, z) \le -\sum_{i=1}^{p} \tilde{z}_i^2 + \frac{\beta}{\bar{\alpha}} \sum_{i=2}^{p} \sum_{j=1}^{i-1} \frac{\sqrt{|\lambda_i^k|}}{\sqrt{|\lambda_j^k|}} \,|\tilde{z}_i|\, |\tilde{z}_j|\,.$$

Now, choose the λ_i^k's, $k = 1, \dots, n-1$; $i = 1, \dots, p$, such that the following inequalities hold:

$$\begin{aligned} &\frac{\beta}{\bar{\alpha}} \frac{\sqrt{|\lambda_i^k|}}{\sqrt{|\lambda_1^k|}} < 2, \qquad 2 \le i \le p, \\ &\frac{\beta}{\bar{\alpha}} \frac{\sqrt{|\lambda_i^k|}}{\sqrt{|\lambda_2^k|}} < 2, \qquad 3 \le i \le p, \\ &\qquad \vdots \\ &\frac{\beta}{\bar{\alpha}} \frac{\sqrt{|\lambda_p^k|}}{\sqrt{|\lambda_{p-1}^k|}} < 2. \end{aligned}$$

Set $\eta = \min_{i>j}\{(\beta/\bar{\alpha})\sqrt{|\lambda_i^k|}/\sqrt{|\lambda_j^k|}\}$, we obtain,

$$\begin{aligned} P(t,z) &\leq -\left(1-\frac{\eta}{2}\right)\left(\tilde{z}_1^2+\cdots+\tilde{z}_k^2\right) \\ &\leq -\mu_k\left(z_1^2+\cdots+z_k^2\right), \end{aligned}$$

where $\mu_k = (1-\eta/2)\min_{1\leq i\leq p}\{2\bar{\alpha}|\lambda_i^k|\}^{-1}$.

Clearly, the construction of S_k only requires the knowledge of $\bar{\alpha}$, $\bar{\beta}$, and β.

This ends the proof of Lemma 1. ∎

In the sequel, we will use the following notations:

- $$\mathcal{A}_k(t) = \begin{bmatrix} 0 & A_1(t) & 0 & & 0 \\ \vdots & & A_2(t) & & \\ 0 & & & \ddots & A_{k-1}(t) \\ 0 & \dots & & 0 & 0 \end{bmatrix}; \qquad k=2,\dots,n, \tag{20}$$

where the A_i's are defined as in (15); $i=1,\dots,k-1$.

- $C_k = [I_p, 0\dots,0]$ is the $p\times(kp)$ matrix.

- $$\mathcal{S}_k = \begin{pmatrix} S_{11} & \nu_1 S_1 & 0 & \dots & 0 \\ \nu_1 S_1^\top & S_{22} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \nu_{k-1}S_{k-1} \\ 0 & \dots & 0 & \nu_{k-1}S_{k-1}^\top & S_{kk} \end{pmatrix}; \qquad k=2,\dots,n, \tag{21}$$

where $\nu_i>0$, $i=1,\dots,k-1$, the S_i; $i=1,\dots,k-1$, are the diagonal negative definite matrices given by Lemma 1 and the S_{ii}; $i=1,\dots,k$, are symmetric positive definite (S.P.D.) constant matrices.

We now state the following.

LEMMA 2. *There exists $\mathcal{S}_n$ a $pn\times pn$ S.P.D. constant matrix of the form (21); $\exists\,\eta_n>0$ such that*

$$\mathcal{A}_n^\top(t)\mathcal{S}_n + \mathcal{S}_n\mathcal{A}_n(t) - C_n^\top C_n \leq -\eta_n I_{pn}, \qquad \forall\, t\geq 0, \tag{22}$$

where $\mathcal{A}_n(t)$ is defined in (20) and I_{pn} is the $p\times n$ identity matrix.

PROOF OF LEMMA 2. we will proceed by induction on the number of blocs, n.

Set

$$Q_k(t) = \mathcal{A}_k^\top(t)\mathcal{S}_k + \mathcal{S}_k\mathcal{A}_k(t) - C_k^\top C_k, \qquad \text{for } k=2,\dots,n.$$

For $n=2$, a simple calculation gives

$$Q_2 = \begin{pmatrix} -I_p & S_{11}A_1 \\ A_1^\top S_{11} & \nu_1\left(A_1^\top S_1 + S_1^\top A_1\right) \end{pmatrix}.$$

Now, let S_{11} be an arbitrary constant S.P.D. matrix and let $x = \binom{x^1}{x^2}$; $x^1, x^2 \in R^p$, we obtain

$$\begin{aligned} x^\top Q_2 x &= -\left\|x^1\right\|^2 + 2\left(x^2\right)^\top A_1^\top(t)S_1x^2 + 2\left(x^1\right)^\top S_{11}A_1(t)x^2 \\ &\leq -\left\|x^1\right\|^2 - \nu_1\mu_1\left\|x^2\right\|^2 + 2\gamma\beta\left\|x^1\right\|\left\|x^2\right\|, \end{aligned}$$

where $\gamma = \|S_{11}\|$ and β is the Lipschitz constant.

Set $\tilde{x} = \begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{pmatrix}$ where $\tilde{x}^1 = x^1$ and $\tilde{x}^2 = x^2\sqrt{\nu_1\mu_1}$. We obtain

$$\begin{aligned}
\tilde{x}^\top Q_2 \tilde{x} &\leq -\left\|\tilde{x}^1\right\|^2 - \left\|\tilde{x}^2\right\|^2 + 2\frac{\gamma\beta}{\sqrt{\nu_1\mu_1}}\left\|\tilde{x}^1\right\|\left\|\tilde{x}^2\right\| \\
&\leq -\left(1 - \frac{\gamma\beta}{\sqrt{\nu_1\mu_1}}\right)\|\tilde{x}\|^2 \\
&\leq -\eta_2\|\tilde{x}\|^2,
\end{aligned}$$

where $\eta_2 = (1 - \gamma\beta/\sqrt{\mu_1\nu_1}) > 0$ as soon as $\nu_1 > \gamma^2\beta^2/\mu_1$.

Finally, it suffices to choose S_{22} such that the matrix $\mathcal{S}_2$ is S.P.D.

Now, assume that (22) holds for $n-1$. It means that there exists a constant S.P.D. matrix $\mathcal{S}_{n-1}$ s.t. for every trajectory of (13), we have

$$\mathcal{A}_n^\top(t)\mathcal{S}_{n-1} + \mathcal{S}_{n-1}\mathcal{A}_{n-1}(t) - C_{n-1}^\top C_{n-1} \leq -\eta_{n-1}I_{(n-1)p},$$

where $\eta_{n-1} > 0$ is a constant.

Let us show (22) for n. Consider the following matrix:

$$\mathcal{S}_n = \left(\begin{array}{ccccc|c}
 & & & & & 0 \\
 & & & & & \vdots \\
 & & \mathcal{S}_{n-1} & & & 0 \\
 & & & & & \nu_{n-1}S_{n-1} \\
0 & \cdots & \cdots & 0 & \nu_{n-1}S_{n-1} & S_{n,n}
\end{array}\right),$$

where $S_{n,n}$ is a S.P.D. matrix which will be specified later, and $\mathcal{S}_{n-1}$ is given by the induction hypothesis.

Set

$$Q_n = \mathcal{A}_n^\top(t)\mathcal{S}_n + \mathcal{S}_n\mathcal{A}_n(t) - C_n^\top C_n.$$

A simple calculation gives:

$$Q_n = \left(\begin{array}{cccc|c}
 & & & & 0 \\
 & Q_{n-1} & & & \vdots \\
 & & & & 0 \\
 & & & & \nu_{n-2}S_{n-2}A_{n-1} \\
 & & & & S_{n-1,n-1}A_{n-1} \\
0 \quad \cdots \quad 0 & & \nu_{n-2}A_{n-1}^\top S_{n-2}^\top & A_{n-1}^\top S_{n-1,n-1} & \nu_{n-1}\left(A_{n-1}^\top S_{n-1} + S_{n-1}^\top A_{n-1}\right)
\end{array}\right),$$

where

$$Q_{n-1} = \mathcal{A}_{n-1}^\top\mathcal{S}_{n-1} + \mathcal{S}_{n-1}\mathcal{A}_{n-1} - C_{n-1}^\top C_{n-1}.$$

Set $x = \begin{pmatrix} X^{n-1} \\ x^n \end{pmatrix}$, where $X^{n-1} = \begin{pmatrix} x^1 \\ \vdots \\ x^{n-1} \end{pmatrix}$ and $x^i \in \mathbb{R}^p$, we get

$$\begin{aligned}
x^\top Q_n x = &-\left\|x^1\right\|^2 + 2X^{n-1T}\mathcal{A}_{n-1}^\top(t)\mathcal{S}_{n-1}X^{n-1} + 2\nu_{n-1}x^{nT}A_{n-1}^\top(t)S_{n-1}x^n \\
&+ 2\nu_{n-2}x^{nT}S_{n-2}A_{n-1}(t)x^{n-2} + 2x^{nT}S_{n-1,n-1}A_{n-1}(t)x^{n-1} \\
\leq &-\eta_{n-1}\left\|X^{n-1}\right\|^2 - \nu_{n-1}\mu_k\|x^n\|^2 + \delta_1\left\|x^{n-2}\right\|\|x^n\| + \delta_2\left\|x^{n-1}\right\|\|x^n\|,
\end{aligned}$$

where $\delta_1 = 2\nu_{n-2}\|S_{n-2}\|\beta$ and $\delta_2 = 2\|S_{n-1,n-1}\|\beta$ according to (18).

Thus,

$$x^\top Q_n x \leq -\eta_{n-1} \left\|X^{n-1}\right\|^2 - \nu_{n-1}\mu_{n-1} \left\|x^n\right\|^2 + \delta \left\|X^{n-1}\right\| \left\|x^n\right\|, \tag{23}$$

where $\delta = \max(\delta_1, \delta_2)$.

Proceeding as above, it is easy to see that for ν_{n-1} high enough, there exists $\eta_n > 0$ such that

$$x^\top Q_n x \leq -\eta_n \|x\|^2.$$

In a similar way as above, we can choose $S_{n,n}$ S.P.D. such that $\mathcal{S}_n$ remains S.P.D.

This ends the proof of Lemma 2. ∎

Our candidate observer takes the following form:

$$\dot{\hat{x}} = F(u, \hat{x}) - \theta \Delta_\theta \mathcal{S}^{-1} C^\top (C\hat{x} - y). \tag{24}$$

Here, $\mathcal{S} \triangleq \mathcal{S}_n$ given by Lemma 2, $C \triangleq C_n = [I_p, 0, \dots, 0]$, $\hat{x} = \begin{bmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \vdots \\ \hat{x}^n \end{bmatrix} \in R^{pn}$, u and y are, respectively, the input and the output of system (13), $\Delta_\theta = \operatorname{diag}(I_p, \theta I_p, \theta^2 I_p, \dots, \theta^{n-1} I_p)$ for some $\theta > 0$.

We then state our main theorem.

THEOREM 2. *Under Assumptions (A′1), (A′2), there exists $\theta_0 > 0$ such that*

$$\forall\, \theta > \theta_0; \qquad \forall\, u \in U; \qquad \exists\, \lambda_\theta > 0; \qquad \exists\, \mu_\theta > 0; \qquad \forall\, x(0) \in K; \qquad \forall\, \hat{x}(0) \in \mathbb{R}^{pn};$$

$$\|\hat{x}(t) - x(t)\| \leq \lambda_\theta e^{-\mu_\theta t} \|\hat{x}(0) - x(0)\|,$$

where $x(t)$ is the trajectory of (13) associated to the initial state $x(0)$ and the input u, $\hat{x}(t)$ is any trajectory of system (24) with input u and output y.

PROOF OF THEOREM 2. Set

- $$e = \begin{bmatrix} e^1 \\ e^2 \\ \vdots \\ e^n \end{bmatrix} = \hat{x} - x, \text{ where } e^k = \hat{x}^k - x^k = \begin{bmatrix} e_1^k \\ e_2^k \\ \vdots \\ e_p^k \end{bmatrix}; \qquad k = 1, \dots, n.$$
- $$\underline{x}^k = \begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix}; \qquad k = 1, \dots, n.$$
- $$\underline{x}_i^k = \begin{pmatrix} x_1^k \\ \vdots \\ x_i^k \end{pmatrix}; \qquad k = 1, \dots, n; \quad i = 1, \dots, p.$$
- $$\delta f_i^k(u, \hat{x}, x) = f_i^k\left(u, \underline{\hat{x}}^k, \underline{x}_i^{k+1}\right) - f_i^k\left(u, \underline{x}^k, \underline{x}_i^{k+1}\right); \quad k = 1, \dots, n-1; \quad i = 1, \dots, p.$$
- $$\delta f_i^n(u, \hat{x}, x) = f_i^n(u, \hat{x}) - f_i^n(u, x); \qquad i = 1, \dots, p. \tag{25}$$
- $$\delta f = \begin{pmatrix} \delta f^1(u, \hat{x}, x) \\ \vdots \\ \delta f^n(u, \hat{x}, x) \end{pmatrix}, \qquad \text{with } \delta f^k(u, \hat{x}, x) = \begin{pmatrix} \delta f_1^k(u, \hat{x}, x) \\ \vdots \\ \delta f_p^k(u, \hat{x}, x) \end{pmatrix}; \quad k = 1, \dots, n.$$

According to the triangular structure of the f_i^k's (given in (11)) and to the above notations, we obtain

- for $1 \le k \le n-1$, $1 \le i \le p$:

$$f_i^k(u,\hat{x}) - f_i^k(u,x) = f_i^k\left(u, \hat{\underline{x}}^k, \hat{\underline{x}}_i^{k+1}\right) - f_i^k\left(u, \hat{\underline{x}}^k, \underline{x}_i^{k+1}\right) + \delta f_i^k. \qquad (26)$$

Now, using the main value theorem and Assumption (A′2), we obtain

$$f^k(u,\hat{x}) - f^k(u,\hat{x}) = A_k(t)e^{k+1}(t) + \delta f^k(u,\hat{x}(t),x(t)); \qquad k = 1,\dots,n-1, \qquad (27)$$

where $A_k(t)$ is given by (15) and its diagonal coefficients satisfy (16).

Combining (27) and (25), we get

$$\dot{e} = \left(\mathcal{A}(t) - \theta\Delta_\theta \mathcal{S}^{-1}C^\top C\right)e + \delta f(u,\hat{x}(t),x(t)),$$

where

$$\mathcal{A}(t) = \begin{bmatrix} 0 & A_1(t) & 0 & & 0 \\ \vdots & & A_2(t) & & \\ 0 & & & \ddots & A_{n-1}(t) \\ 0 & \dots & & 0 & 0 \end{bmatrix}.$$

Now, set $\bar{e} = \Delta_\theta^{-1}e$. A simple calculation gives:

$$\dot{\bar{e}} = \theta\left(\mathcal{A}(t) - \mathcal{S}^{-1}C^\top C\right)\bar{e} + \Delta_\theta^{-1}\delta f(u,\hat{x}(t),x(t)).$$

Consider the quadratic positive definite function $V(\bar{e}) = \bar{e}^\top\mathcal{S}\bar{e}$, we obtain

$$\begin{aligned} \dot{V} &= 2\bar{e}^\top\mathcal{S}\dot{\bar{e}} \\ &= 2\theta\bar{e}^\top\left(\mathcal{S}\mathcal{A}(t) - C^\top C\right)\bar{e} + 2\bar{e}^\top\mathcal{S}\Delta_\theta^{-1}\delta f(u,\hat{x}(t),x(t)). \end{aligned} \qquad (30)$$

According to Lemma 2, we obtain

$$\dot{V} \le -\theta\eta_n\bar{e}^\top\bar{e} + 2\|\mathcal{S}\bar{e}\|\left\|\Delta_\theta^{-1}\delta f(u,\hat{x}(t),x(t))\right\|.$$

Now, take $\theta \ge 1$, then, because of the triangular structure and the Lipschitz assumption of the functions f_i^ks, we have

$$\left\|\Delta_\theta^{-1}\delta f(u,\hat{x}(t),x(t))\right\| \le c\|\bar{e}\|$$

for some constant c which does not depend on θ.

Thus,

$$\dot{V} \le (-\theta\eta_n + 2c\|\mathcal{S}\|)\|\bar{e}\|^2 \qquad (32)$$

$$\le \left(-\theta\frac{\eta_n}{\lambda_{\min}(\mathcal{S})} + 2c\frac{\|\mathcal{S}\|}{\lambda_{\min}(\mathcal{S})}\right)V. \qquad (33)$$

Now, choose $\theta > 2c\|\mathcal{S}\|/\eta_n$.

This ends the proof of our theorem. ∎

4. CONCLUSION

A simple observer for a special class of multi-output nonlinear systems has been derived. The main characteristics of the proposed observer lies in the fact that its gain is constant and does not necessitate the resolution of any dynamical system. Indeed, a simple and systematic procedure allowing the gain construction has been presented. A similar design can be made for a more general multi-output class of nonlinear systems. Current research is underway to address this problem.

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